The equilibrium problem and applications

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Introduction

One of the most important problems in nonlinear analysis is the so called *(scalar) equilibrium problem* (abbreviated (EP)), which can be formulated as follows. Let A and B be two nonempty sets and $f : A \times B \to \mathbb{R}$ a given function. The problem consists on finding an element $\bar{a} \in A$ such that

$$f(\bar{a}, b) \geq 0, \quad \text{for all } b \in B. \tag{1}$$

The element $\bar{a}$ satisfying (1) is called *equilibrium point* of f on $A \times B$.

(EP) has been extensively studied in recent years (e.g. [27], [28], [29], [30], [31], [82], [83], [84], [88] and the references therein). Apart from its theoretical interest, important problems arising from economics, mechanics, electricity and other practical sciences motivate the study of (EP). Equilibrium problems include, as particular cases, *scalar and vector optimization problems, saddle point (minimax) problems, variational inequalities, Nash equilibria problems, complementarity problems, fixed point problems*, etc.

As far as we know the term "equilibrium problem" was attributed in [31], but the problem itself has been investigated more than twenty years before in a paper of Ky Fan [69] in connection with the so called "intersection theorems" (i.e., results stating the nonemptiness of a certain family of sets). Ky Fan considered (EP) in the special case $A = B$ a compact convex subset of a Hausdorff topological vector space and termed it "minimax inequality". Within short time (in the same year) Brézis, Nirenberg and Stampacchia [50] improved Ky Fan's result, extending it to a not necessarily compact set, but assuming instead a so-called "coercivity condition", which is automatically satisfied when the set is compact.

Recent results on (EP) emphasizing the existence of solutions can be found in [27], [28], [29], [120], and many other papers. New necessary (and in some cases also sufficient) conditions for existence of solutions in infinite dimensional spaces were proposed in [83], and later on simplified and further analyzed in [82].

The first fundamental concept in well posedness area is inspired by the classical idea of J. Hadamard in 1922, which goes back to the beginning of the previous century. It requires the existence and uniqueness of the optimal solution together with continuous dependence on the problems data.

In the early sixties A. Tikhonov introduced another concept of well posedness imposing convergence of every minimizing sequence to the unique minimum point. Its relevance to the approximate solution of optimization problems is clear.

Let a scalar optimization problem (D, h)

$$\min h(a), \quad a \in D$$

where $h : D \to \mathbb{R}$, and D is a nonempty set. The problem is Tikhonov well posed if and only if there exists exactly one $a_0 \in D$ such that $h(a_0) \leq h(a)$ for all $a \in D$ and

$$h(a_n) \to h(a_0)$$

implies $a_n \to a_0$.

Example 0.0.1. *Let $D = \mathbb{R}^n$ and $h(a) = |a|$ (taking any norm).*
Then $0 = \operatorname{argmin}(D, h)$ and cleary (D, h) is Tikhonov well posed.

Example 0.0.2. *Let $D = \mathbb{R}$ and*

$$h(a) = \begin{cases} a & \text{for} \quad a > 0 \\ |a+1| & \text{for} \quad a \leq 0 \end{cases},$$

the problem (D, h) is not Tikhonov well posed (in this case we say that the problem is Tikhonov ill posed). Ineed, the only minimum point is $a_0 = -1$, but the minimizing sequence $a_n = 1/n$ does not converge to a_0.

Dattoro in [59] says that the duality is a powerful and widely employed tool in applied mathematics for a number of reasons. First, the dual program is always convex even if the primal is not. Second, the number of variables in the dual is equal to the number of constraints in the primal which is often less than the number of variables in the primal program. Third, the maximum value achieved by the dual problem is often equal to the minimum of the primal.

This work is organized as follows. First we recall some definitions, which help the reader to understand easily the following parts.

The second chapter is based on the equilibrium problem and its generalizations. We present some existence results of solutions for the scalar and vector equilibrium problems. In recent years the vector and multifunction form of the equilibrium problem has been studied extensively (see, e.g., [[51], [80]). These problems can be formulated as follows. Let A be a nonempty subset of a topological vector space X, B a nonempty set, Z a topological vector space, $C \subset Z$ a convex and solid cone, and $f : A \times B \to Z$ be a vector-valued function. The weak vector equilibrium problem is

$$\text{find } \bar{a} \in A \text{ such that } f(\bar{a}, b) \notin -\operatorname{int} C \text{ for all } b \in B. \tag{2}$$

In the final section of the chapter we extend the results from the vector equilibrium problems to the so-called weak multifunction equilibrium problems. If $f : A \times B \to 2^Z$, one way to define the weak multifunction equilibrium problem is th following:

$$\text{find } \bar{a} \in A \text{ such that } f(\bar{a}, b) \nsubseteq -\operatorname{int} C \text{ for all } b \in B. \tag{3}$$

Observe that this problem reduces to weak vector equilibrium problem when f is single valued. We give two existence theorems and two corollaries for the weak multifunction equilibrium problem.

Chapter three is devoted to well posedness for different equilibrium problems. We establish the relation between Tikhonov well posedness for equilibrium problems and Tikhonov well posedness for noncooperative games, then prove the equivalence of this type of well posedness to equilibrium problems and noncooperative games. Using the results, in the second part we deduce the relation between diameter properties. In the second part of this chapter we extend some results obtained by Bianchi, Kassay and Pini in [26] to the strong vector equilibrium problem. Also, we study the weak vector equilibrium problem and assert the definitions for B-well posedness and M-well posedness to the weak case. The relationship between these type of well posedness is established, and we give sufficient conditions for the equivalence between well posedness notions.

In the last part of this work we discuss some applications. Chapter five is based on the noncooperative game and the cooperative games. First, we present the well known two-person zero sum noncooperative games and saddle points, offering examples. Furthermore, we show how a cooperative game can be obtained from a noncooperative game (Battle of the Sexes). Throughout the following section applications of cooperative games such as “A production economy with landowners and peasants”, ”An exchange economy with traders of two types“, “The airport game”, “The bankruptcy game”, “Cooperative water resource development in Japan”, and the “Simple game“ are presented. In most of these cases the elements of the player set represent real persons, e.g., landowners and peasants, traders, creditors or voters, or the player set can also consist of objectives as in the well-known TVA cases, airport landings by planes or agricultural associations and city water services.

In the last section we deduce the dual representation between characteristic function and indirect function of transferable utility games.

Finally we give a closedness type regularity condition that ensures the maximal monotonicity of the generalized sum $S + A^*TA$ involving strongly-representable monotone operators, and, we show that our condition is weaker than those mentioned above.we give an useful application for the stable strong duality involving the function $f+g\circ A$, where f and g are proper, convex and lower semicontinuous functions, and A is a linear and continuous operator. We also introduce some generalized inf-convolution formulas, and establish some result concerning on their Fenchel conjugate. In the last part, some particular instances, to which the general results on the maximal monotonicity of $S + A^*TA$ give rise, are considered.

The author's contributions to this thesis are based on five papers, four of them written in collaboration. One of them paper [52] concerning weak multifunction equilibrium problems appeared in The Special Volume in Honour of Boris Mordukhovich, Springer Optimization and its Application in 2010, the other [47] published online in Set-Valued and Variational Analysis in 2011, the other three [46], [45], [78] are submitted to ISI journals.

Our original results are formulated in the following definitions, theorems, propositions and corollaries:

Chapter 2: Lemma 2.3.46, Definition 2.3.47, Theorem 2.3.50, Corollary 2.3.49, Corollary 2.3.50.

Chapter 3: Theorem 3.1.9, Theorem 3.1.20, Proposition 3.2.11, Proposition 3.2.12, Definition 3.2.13, Remark 3.2.15, Definition 3.2.16, Proposition 3.2.17, Proposition 3.2.18.

Chapter 4: Remark 4.5.33, Theorem 4.5.35, Corollary 4.5.36, Theorem 4.5.38, Corollary 4.5.39, Theorem 4.5.41, Corollary 4.5.42, Theorem 4.5.43, Corollary 4.5.44, Theorem 4.5.45 Remark 4.5.46, Remark 4.5.47, Remark 4.5.48.

Chapter 5: Theorem 5.2.61, Remark 5.2.63, Theorem 5.2.64, Remark 5.2.65, Corollary 5.2.66, Corollary 5.2.67, Theorem 5.2.68, Remark 5.2.69, Corollary 5.2.70, Theorem 5.3.71, Theorem 5.3.73, Corollary 5.4.74, Corollary 5.4.75, Corollary 5.4.76, Corollary 5.4.77.

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Keywords: weak multifunctuion equilibrium problem, tikhonov well posedness, B-well posedness, M-well posedness, weak vector equilibrium problem, noncooperative games, cooperative games, characteristic function, indirect function, trasferable utility games, monotone operator, strongly-representable operator, representative function, generalized sum.

Chapter 1

Preliminaries

This chapter presents the mathematical notions used throughout the thesis. The definitions of the concepts widely utilized in the field are given here, as well as the remarks and propositions related to these.

Chapter 2

Equilibrium problem

2.1 Scalar equilibrium problem

Let A and B be two nonempty sets and $\varphi : A \times B \to \mathbb{R}$ a given function. The scalar equilibrium problem consists on

$$(EP) \qquad \text{finding } \bar{a} \in A \text{ such that } \varphi(\bar{a}, b) \geq 0 \text{ for all } \; b \in B.$$

We present some existence results of solutions for (EP). A general existence result for the problem (EP) has been established by Kassay and Kolumbán also in [89], where instead of the convexity (concavity) assumptions upon the function f, certain kind of generalized convexity (concavity) assumptions are supposed.

Theorem 2.1.3. *Let A be a compact topological space, let B be a nonempty set, and let $f : A \times B \to \mathbb{R}$ be a given function such that*

(i) *for each $b \in B$, the function $\varphi : A \to \mathbb{R}$ is usc;*

(ii) *for each $a_1, ..., a_m \in A$, $b_1, ..., b_k \in B$, $\lambda_1, ..., \lambda_m \geq 0$ with $\sum_{i=1}^m \lambda_i = 1$, the inequality*

$$\min_{1 \leq j \leq k} \sum_{i=1}^{m} \lambda_i f(a_i, b_j) \leq \sup_{a \in A} \min_{1 \leq j \leq k} f(a, b_j)$$

holds;

(iii) *For each $b_1, ..., b_k \in B$, $\mu_1, ..., \mu_k \geq 0$ with $\sum_{j=1}^k \mu_j = 1$, one has*

$$\sup_{a \in A} \sum_{j=1}^{k} \mu_j f(a, b_j) \geq 0.$$

Then the equilibrium problem (EP) admits a solution.

2.2 Vector equilibrium problem

If the scalar function φ is replaced by a vector-valued function, say $\varphi : A \times B \to Z$ a given function, where A and B are two nonempty sets, Z is a topological vector space, partially ordered by the convex cone $C \subseteq Z$ with $\text{int}C \neq \emptyset$, one may consider the so-called *vector equilibrium problem* in two ways:

$$(VEP) \qquad \text{find } \bar{a} \in A \text{ such that } \varphi(\bar{a}, b) \notin -C\setminus\{0\} \text{ for all } \ b \in B$$

and

$$(WVEP) \qquad \text{find } \bar{a} \in A \text{ such that } \varphi(\bar{a}, b) \notin -\text{int}\, C \text{ for all } \ b \in B.$$

The first problem is called *strong equilibrium problem*, while the second one is called *weak equilibrium problem.*

Let A be a nonempty subset of X, B a nonempty set, and let $\varphi : A \times B \to Z$. The next result provides sufficient condition for the existence of solutions of (WVEP).

Theorem 2.2.4. *[51] Let A be a compact set and let $\varphi : A \times B \to Z$ be a function such that*

(i) for each $b \in B$, the function $\varphi(\cdot, b) : A \to Z$ is usc on A;

(ii) for each $a_1, a_2, \ldots, a_m \in A$, $\lambda_1, \lambda_2, \ldots, \lambda_m$ geq0 with $\sum_{i=1}^m \lambda_i = 1$, $b_1, \ldots, b_n \in B$ there exists $u^* \in C^*\setminus\{0\}$ such that

$$\min_{1 \le j \le n} \sum_{i=1}^{m} \lambda_i u^*(\varphi(a_i, b_i)) \le \sup_{a \in A} \min_{1 \le j \le n} u^*(\varphi(a, b_j));$$

(iii) for each $b_1, \ldots, b_n \in B$ and $z_1^*, \ldots, z_n^* \in C^*$ not all zero one has

$$\sup_{a \in A} \sum_{j=1}^{n} z_j^*(\varphi(a, b_j)) \ge 0.$$

Then the equilibrium problem (WVEP) *admits a solution.*

2.3 Multifunction equilibrium problem

Let A be a nonempty subset of a real topological vector space X, B a nonempty set, Z a normed space, $C \subseteq Z$ a convex and solid cone, and let $\varphi : A \times B \to 2^Z$ be a multifunction. We study the following *weak multifunction equilibrium problem*:

$$(WWMEP) \qquad \text{find } \bar{a} \in A \text{ such that } \varphi(\bar{a}, b) \nsubseteq -\operatorname{int} C \text{ for all } b \in B.$$

By $\mathcal{C}(Z)$ we denote the set of all compact subsets of the space Z.

We need the following technical result whose proof is based on a separation theorem in infinite dimensional spaces.

Lemma 2.3.5. *(A. Capata, G. Kassay,* **B. Mosoni** *[52]) Let $\varphi : A \times B \to \mathcal{C}(Z)$ be a multifunction such that*

(i) if the system $\{U_{b,k} \mid b \in B,\, k \in \operatorname{int} C\}$ covers A, then it contains a finite subcover, where

$$U_{b,k} = \{a \in A | \varphi(a, b) + k \subseteq -\operatorname{int} C\};$$

(ii) for each $a_1, \ldots, a_m \in A$, $\lambda_1, \ldots, \lambda_m \geq 0$ with $\sum_{i=1}^m \lambda_i = 1$, $b_1, \ldots, b_n \in B$, for all $d_j^i \in \varphi(a_i, b_j)$ where $i \in \{1, \ldots, m\}$ and $j \in \{1, \ldots, n\}$ there exists $u^* \in C^* \setminus \{0\}$ such that

$$\min_{1 \leq j \leq n} \sum_{i=1}^m \lambda_i u^*(d_j^i) \leq \sup_{a \in A} \min_{1 \leq j \leq n} \ \max u^*(\varphi(a, b_j)),$$

where $\max u^*(\varphi(a, b_j))$ is the greatest element of the compact set $u^*(\varphi(a, b_j)) \subseteq \mathbb{R}$;

(iii) for each $b_1, \ldots, b_n \in B$ and $z_1^*, \ldots, z_n^* \in C^*$ not all zero

$$\sup_{a \in A} \sum_{j=1}^n \ \max z_j^*(\varphi(a, b_j)) \geq 0.$$

Then the equilibrium problem $(WWMEP)$ admits a solution.

Following the definition of C-subconvexlikeness we introduce a new convexity notion.

Definition 2.3.6. *(A. Capata, G. Kassay,* **B. Mosoni** *[52]) Let $T : X \times Y \to 2^Z$ be a multifunction, $C \subset Z$ a convex and solid cone. T is said to be C-subconvexlike in its first variable if for each $\theta \in \operatorname{int} C$, x_1, $x_2 \in X$ and $t \in (0, 1)$ there exists an $x_3 \in X$ such that*

$$\theta + tT(x_1, y) + (1 - t)T(x_2, y) \subset T(x_3, y) + \operatorname{int} C \textit{ for all } y \in Y.$$

We say that T is C-subconcavelike in its first variable if $-T$ is C-subconvexlike in its first variable.

The next result provides sufficient conditions for the existence of $(WWMEP)$ by means of convexity and continuity assumptions.

Theorem 2.3.7. *(A. Capata, G. Kassay,* **B. Mosoni** *[52]) Let A be a compact set and $\varphi : A \times B \to \mathcal{C}(Z)$ such that:*
(i) $\varphi(\cdot, b)$ is upper $-C$-continuous for all $b \in B$;
(ii) φ is C-subconcavelike in its first variable;
(iii) for each $b_1, \ldots, b_n \in B$ and $z_1^*, \ldots, z_n^* \in C^*$ not all zero yields

$$\sup_{a \in A} \sum_{j=1}^{n} max\, z_j^*(\varphi(a, b_j)) \geq 0.$$

Then the equilibrium problem $(WWMEP)$ admits a solution.

Now, let us consider the particular case: $Z = \mathbb{R}$ and $C = \mathbb{R}_+$. Then $\varphi : A \times A \to 2^{\mathbb{R}}$ and $(WWMEP)$ becomes:

$$(MEP) \text{ find } \bar{a} \in A \text{ such that } \varphi(\bar{a}, b) \nsubseteq -\operatorname{int} \mathbb{R}_+ \text{ for all } b \in A.$$

For this particular case, using the previous results we obtain the following.

Corollary 2.3.8. *(A. Capata, G. Kassay,* **B. Mosoni** *[52]) Let $\varphi : A \times B \to \mathcal{C}(\mathbb{R})$ be a multifunction such that*
(i) if the system $\{U_{b,k} \mid b \in B,\, k > 0\}$ covers A, then it contains a finite subcover, where

$$U_{b,k} = \{a \in A | \varphi(a, b) + k \subseteq -\operatorname{int} \mathbb{R}_+\};$$

(ii) for each $a_1, \ldots, a_m \in A$, $\lambda_1, \ldots, \lambda_m \geq 0$ with $\sum_{i=1}^m \lambda_i = 1$, $b_1, \ldots, b_n \in B$, for all $d_j^i \in \varphi(a_i, b_j)$ where $i \in \{1, \ldots, m\}$ and $j \in \{1, \ldots, n\}$

$$\min_{1 \leq j \leq n} \sum_{i=1}^{m} \lambda_i d_j^i \leq \sup_{a \in A} \min_{1 \leq j \leq n} \max \varphi(a, b_j);$$

(iii) for each $b_1, \ldots, b_n \in B$ and $z_1^*, \ldots, z_n^* \geq 0$ not all zero

$$\sup_{a \in A} \sum_{j=1}^{n} \max\, z_j^*(\varphi(a, b_j)) \geq 0.$$

Then the equilibrium problem (MEP) admits a solution.

Corollary 2.3.9. *(A. Capata, G. Kassay,* **B. Mosoni** *[52]) Let A be a compact set and $\varphi : A \times B \to \mathcal{C}(\mathbb{R})$ such that:*
(i) $\varphi(\cdot, b)$ is upper $-\mathbb{R}_+$-continuous for all $b \in B$;
(ii) φ is $\mathbb{R}_+$-subconcavelike in its first variable;
(iii) for each $b_1, \ldots, b_n \in B$ and $z_1^*, \ldots, z_n^* \geq 0$ not all zero yields

$$\sup_{a \in A} \sum_{j=1}^{n} max\, z_j^*(\varphi(a, b_j)) \geq 0.$$

Then the equilibrium problem (MEP) admits a solution.

Let A be a nonempty, closed and convex subset of a real locally convex space and suppose that $\varphi(a, b)$ is a compact subset of $\mathbb{R}$ for each $a, b \in A$. We observe that (MEP) is equivalent to the problem:

$$\text{find } \bar{a} \in A \text{ such that } \max \varphi(\bar{a}, b) \geq 0 \;\text{ for all } b \in A,$$

or, in order words:

$$(EP) \text{ find } \bar{a} \in A \text{ such that } \psi(\bar{a}, b) \geq 0 \;\text{ for all } b \in A,$$

where $\psi : X \times X \to \mathbb{R} \cup \{+\infty\}$, with $A \times A \subseteq \operatorname{dom} f$, defined by $\psi(a, b) = \max \varphi(a, b)$ for all $a, b \in A$. Further suppose that $\max \varphi(a, a) = 0$ for all $a \in A$. Let $a \in X$. According to [4], (EP) can be reduced to the optimization problem

$$P(a) \qquad \inf_{b \in A} \psi(a, b).$$

Is easy to check that $\bar{a} \in A$ is a solution of (EP) if and only if it is a solution of $P(\bar{a})$.

Chapter 3

Well posed equilibrium problem

3.1 Tikhonov well posedness

Given a nonempty set D and a function $F : D \times D \to \mathbb{R}$, the problem of interest, called equilibrium problem (EP) consists of finding an element $a \in D$ such that

$$F(\bar{a}, b) \geq 0, \text{ for every } b \in D. \tag{1}$$

Let F be a given function such that $F(a, a) = 0$ for every $a \in D$.

Let the extended-valued gap function $G : D \to [-\infty, +\infty)$ defined by $G(a) = \inf_{b \in D} F(a, b)$, and G is non-positive on the set D, and $G(\bar{a}) = 0$ if and only if $\bar{a}$ is a solution of EP.

Definition 3.1.1. *[25] The equilibrium problem EP is Tikhonov well-posed if*

(i) there exists only one solution $a \in D$ of EP,

(ii) for every sequence $\{a_n\} \subset D$ such that $G(a_n) \to 0$, it is $a_n \to a$.

Definition 3.1.2. *A game G=(X, Y, f, g) is called Tikhonov well posed,*

(i) if there is a unique $(\bar{x}, \bar{y})$ Nash equilibrium and

(ii) every asymptotically Nash equilibrium (x_n, y_n) converges to $(\bar{x}, \bar{y})$.

Now we are able to assert the following result:

Theorem 3.1.3. *(***B. Burjan-Mosoni (Mosoni)** *[45]) Let X, Y Hausdorff topological spaces and $G = (X, Y, f, g)$ the associated two person game with the real valued utility functions f, g.*

The game G is Tikhonov well posed if and only if the equilibrium problem EP($F, X \times Y$) is Tikhonov well posed too, where $F(a, b) = f(x, y) - f(u, y) + g(x, y) - g(x, v)$ for all $a = (x, y) \in X \times Y$ and $b = (u, v) \in X \times Y$.

The relation between the diameters.

Theorem 3.1.4. *(***B. Burjan-Mosoni (Mosoni)*** [45]) If there is a Nash equilibrium for the game $G = (X, Y, f, g)$ and*

$$\lim_{\epsilon \to 0, k \to \infty} \operatorname{diam} \Omega_\epsilon^k = 0,$$

then

$$\operatorname{diam}(\epsilon - \operatorname{argmin}(EP)) \to 0, \ \textit{where } \epsilon \searrow 0.$$

Moreover, the converse is true if $a \to F(a, b)$ is upper semi continuous for every $b \in D$ and every $\epsilon > 0$ and the payoff functions f and g are bounded from above.

3.2 B-well posedness and M-well posedness for vector equilibrium problem

Let X, Y be topological vector spaces with countable bases and C be a closed convex cone in Y with nonempty interior. Given $f : X \times X \to Y$ with property $f(x,x) = 0$, for all $x \in X$, the weak vector equilibrium problem is: find $\bar{x} \in X$ such that

$$f(\bar{x}, y) \notin -\text{int } C, \text{ for all } y \in X \tag{2}$$

We introduce the set valued map $\Phi : X \to 2^Y$ (see also [11]) given by:

$$\Phi(x) = \text{w-}\min_C(f(x, X)), \tag{3}$$

where for any $A \subset Y$, the set of minimal elements is defined as follows: $\text{w-min}_C(A) = \{a' \in A : (A - a') \cap (-\text{int } C) = \emptyset\}$.

The $\bar{x} \in S$ if and only if $0 \in \Phi(\bar{x})$, where we call S the solution set and we will suppose in the sequel that S is nonempty.

Proposition 3.2.1. *The map Φ satisfies the relations:*

1. $\Phi(x) \cap \text{int } C = \emptyset$, *for all* $x \in X$;
2. $\bar{x} \in S \Leftrightarrow 0 \in \Phi(\bar{x})$;
3. $\bar{x} \in S \Rightarrow \Phi(\bar{x}) \cap C \neq \emptyset$;
4. $\bar{x} \in S \Leftrightarrow \Phi(\bar{x}) \cap C' \neq \emptyset$;

where $C' = (\text{int } C) \cup \{0\}$.

Proposition 3.2.2. *If $f(x, y) = F(y) - F(x)$, then $\{x_n\}$ is maximizing if and only if*

$F(x_n) \rightharpoonup_H \text{w-min}_C F(X)$ *i.e., $\{x_n\}$ is a minimizing sequence for the vector optimization problem, according to [115].*

Definition 3.2.3. *We say that the vector equilibrium problem (2) is M-well-posed if:*

(i) there exists at least one solution, i.e., $S \neq \emptyset$;

(ii) for every maximizing sequence, and for every $V_X \in \mathcal{V}_X(0)$, there exists n_0 such that $x_n \in S + V_X$, for every $n \geq n_0$.

In what follows we extend the definition of $\epsilon - \text{argmin}(EP)$ to the weak vector valued case ($\epsilon - \text{argmin}(EP)$).

Definition 3.2.4. *Given* $\epsilon \in C$*, the set*
$S(\epsilon) = \{x \in X : \Phi(x) \cap (C - \epsilon) \neq \emptyset\}$ *is called the* ϵ*-approximate solution set of (2).*

Notice that $S(0) = S$, by definition 3.

Remark 3.2.5. *This definition can be also related to the notion of* ϵ*-weak-minimal solutions* $wQ(\epsilon) = \cup_{y \in \text{w-min}_C F(X)} \{x \in X : F(x) \in y + \epsilon - C\}$*. In case of vector optimization problems, where* $f(x, y) = F(y) - F(x)$*, one trivially shows that* $S(\epsilon) = wQ(\epsilon)$ *for every* $x \in X$*.*

Definition 3.2.6. *We say that the vector equilibrium problem (2) is B-well-posed if*
(i) there exists at least one solution, i.e., $S \neq \emptyset$*;*
(ii) the map $S(\cdot) : C \to 2^X$ *is upper Hausdorff continuous at* $\epsilon = 0$*, i.e., for every* $V_X \in \mathcal{V}_X(0)$ *there exists* $V_Y \in \mathcal{V}_Y(0)$ *such that* $S(\epsilon) \subset S + V_X$ *for every* $\epsilon \in V_Y \cap C$*.*

Proposition 3.2.7. *Any B-well-posed weak vector equilibrium problem is M-well-posed.*

Proposition 3.2.8. *Assume that the weak vector equilibrium problem is M-well-posed and for every* $V_Y \in \mathcal{V}_Y(0)$ *there exists* $\tilde{V}_Y \in \mathcal{V}_Y(0)$ *such that*

$$\Phi(X \backslash cl(S)) \cap (C + \tilde{V}_Y) \subseteq V_Y.$$

Then, the problem is B-well-posed.

Chapter 4

Noncooperative and cooperative games

4.1 Two-person zero sum noncooperative games and saddle points

The saddle point (minimax theorems)

Let X, Y be two nonempty sets and $h : X \times Y \to \mathbb{R}$ be a given function. The pair $(\bar{x}, \bar{y}) \in X \times Y$ is called a *saddle point of* h *on the set* $X \times Y$ if

$$h(x, \bar{y}) \leq h(\bar{x}, \bar{y}) \leq h(\bar{x}, y), \ \forall (x, y) \in X \times Y. \tag{1}$$

Let $A = B = X \times Y$ and let $f : A \times B \to \mathbb{R}$ defined by

$$f(a, b) := h(x, v) - h(u, y), \ \forall a = (x, y), \, b = (u, v). \tag{2}$$

Then each solution of the equilibrium problem (EP) is a saddle point of h, and vice-versa.

The saddle point can be characterized as follows. Suppose that for each $x \in X$ there exists $\min_{y \in Y} h(x, y)$, and for each $y \in Y$ there exists $\max_{x \in X} h(x, y)$. Then we have the following result.

Proposition 4.1.9. *f admits a saddle point on $X \times Y$ if and only if there exist* $\max_{x \in X} \min_{y \in Y} f(x, y)$ *and* $\min_{y \in Y} \max_{x \in X} f(x, y)$ *and they are equal.*

Two-player zero-sum games

Duality in optimization

This (general) problem has many important particular cases: *The optimization problem with inequality and equality constraints.*

This problem has two main cases: The linear programming problem. The conical programming problem.

4.2 Examples for noncooperative games

To underline the importance of (EP) we present in this section some of its various particular cases which have been extensively studied in the literature. The most of them are models of real life problems originated from mechanics, economy, biology, etc.

The convex minimization problem Fixed point problem Complementarity problem Nash equilibria problem in noncooperative games Vector Minimization Problem

4.3 Cooperative games obtained by noncooperative games

Let us recall the "Battle of the Sexes" game where the strategies are given. The corresponding bilosses are given by the matrix

$$L := \begin{pmatrix} (1,4) & (0,0) \\ (0,0) & (4,1) \end{pmatrix}$$

Definition 4.3.10. *Let G_2 be a (noncooperative) two-person game with finite strategy sets S_1 and S_2 and let $L = (L_1, L_2)$ be its biloss operator. Then the corresponding cooperative game is given by the biloss operator*

$$\hat{L} : \Delta^{S_1 \times S_2} \to \mathbb{R} \times \mathbb{R}$$

$$\sum_{i,j} \lambda_{ij}(s_i, \tilde{s}_j) \mapsto \sum_{i,j} \lambda_{ij} L(s_i, \tilde{s}_j)$$

where $\Delta^{S_1 \times S_2} = \{\sum_{i,j} \lambda_{ij}(s_i, \tilde{s_j}) | \sum_{i,j} \lambda_{ij} = 1, \lambda_{ij} \in [0,1]\}$ is the (formal) simplex spanned by the pure strategy pairs $(s_i, \tilde{s_j})$.

Definition 4.3.11. *Given a two person game G_2 and let $\hat{L}$ be the biloss operator of the corresponding cooperative game. A pair of losses $(u,v) \in \mathrm{im}(\hat{L})$ is called jointly sub-dominated by a pair $(u',v') \in \mathrm{im}(\hat{L})$ if $u' \leq u$ and $v' \leq v$ and $(u',v') \neq (u,v)$. The pair (u,v) is called Pareto optimal if it is not jointly sub-dominated.*

Definition 4.3.12. *Given a two person game G_2 and let $\hat{L}$ be the biloss operator of the corresponding cooperative game. The set*

$$B := \{(u,v) \in \mathrm{im}(L) | u \leq u^*, v \leq v^* \text{ and } (u,v) \text{ Pareto optimal } \}$$

is called the bargaining set (sometimes also negotiation set).

4.4 Cooperative games in characteristic function form

Definition 4.4.13. *Let $n \in \mathbb{N}$. A cooperative n-person game in characteristic function form is an ordered pair (N, v), where N is a set of n elements and $v : 2^N \to \mathbb{R}$ is a real-valued set-function on the set 2^N of all subsets of N such that $v(\emptyset) = 0$.*

Elements of the set N are called players and the relevant set-function v the characteristic function of the game. A subset S of the player set N ($S \subset N$) is called a coalition and $v(S)$ the worth of coalition S in the game. In many cases, the elements of the player set N represent real persons, e.g., landowners and peasants, traders, creditors or voters, but the player set can also consist of objectives as in the well known TVA cases, airport landings by planes or agricultural associations and city water services.

4.5 Dual representation between characteristic function and indirect function of TU games

Fix the player set N and its power set $\mathcal{P}(N) = \{S | S \subseteq N\}$ consisting of all the subsets of N (including the empty set $\emptyset$). A *cooperative transferable utility (TU) game* is given by the so-called *characteristic function* $v : \mathcal{P}(N) \to \mathbb{R}$ satisfying $v(\emptyset) = 0$. That is, the TU game v assigns to each coalition $S \subseteq N$ its *worth* $v(S)$ amounting the (monetary) benefits achieved by cooperation among the members of S.

Definition 4.5.14. *([100], page 292) With every n-person TU game $v : \mathcal{P}(N) \to \mathbb{R}$, there is associated the* indirect function $\pi^v : \mathbb{R}^N \to \mathbb{R}$, *given by*

$$\pi^v(\vec{y}) = \max_{S \subseteq N} \{v(S) - \sum_{k \in S} y_k\} \text{ for all } \vec{y} = (y_k)_{k \in N} \in \mathbb{R}^N, \tag{3}$$

where, for every $S \subseteq N$ (including the empty set $\emptyset$), the excess $e^v(S, \vec{y}) = v(S) - \sum_{k \in S} y_k$.

Remark 4.5.15. *(D. Hou, T.S.H. Driessen, A. Meseguer-Artola,* **B. Mosoni** *[78]) With a set $X \subseteq \mathbb{R}^n$ and a function $f : X \to \mathbb{R} \cup \{+\infty, -\infty\}$, there is* *associated its* Fenchel–Moreau conjugate function $f_X^* : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty, -\infty\}$ *defined by*

$$f_X^*(\vec{y}) = sup[< \vec{y}, \vec{x} > -f(\vec{x}) \mid \vec{x} \in X] \quad \text{for all } \vec{y} \in \mathbb{R}^n$$

In the setting of an n-person TU game $v : \mathcal{P}(N) \to \mathbb{R}$ with $v(\emptyset) = 0$, put $X = \{-1_S \in \mathbb{R}^n \mid \quad S \subseteq N\}$ as well as the function $f^v : X \to \mathbb{R}$ given by $f^v(\vec{x}) = -v(S)$ whenever $\vec{x} = -1_S$, then the Fenchel-Moreau conjugate $f_{v,X}^* : \mathbb{R}^n \to \mathbb{R}$ agrees with the indirect function π^v of the form (3).

Theorem 4.5.16. *(D. Hou, T.S.H. Driessen, A. Meseguer-Artola,* **B. Mosoni** *[78]) Let the n-person TU game $v : \mathcal{P}(N) \to \mathbb{R}$ be 1-convex. Then its indirect function $\pi^v : \mathbb{R}^N \to \mathbb{R}$ satisfies the following properties:*

(i) $\pi^v(\vec{y}) = \max[0, \quad v(N) - \sum_{k \in N} y_k]$ *for all* $\vec{y} = (y_k)_{k \in N} \in \mathbb{R}^N$ *with* $y_i \leq b_i^v$ *for all* $i \in N$.

(ii)

$$\begin{aligned} \pi^v(\vec{y}) &= \max[0, \quad v(N \setminus \{\ell\}) - \sum_{k \in N \setminus \{\ell\}} y_k] \\ &= \max[0, \quad v(N) - \sum_{k \in N} y_k + y_\ell - b_\ell^v] \end{aligned}$$

for all $\vec{y} = (y_k)_{k \in N} \in \mathbb{R}^N$ *such that there exists a unique* $\ell \in N$ *with* $y_\ell > b_\ell^v$ *and* $y_i \leq b_i^v$ *for all* $i \in N$, $i \neq \ell$.

Corollary 4.5.17. *(D. Hou, T.S.H. Driessen, A. Meseguer-Artola,* **B. Mosoni** *[78]) For every 1-convex n-person TU game $v : \mathcal{P}(N) \to \mathbb{R}$, the following three statements concerning a payoff vector $\vec{y} = (y_k)_{k \in N} \in \mathbb{R}^N$ are equivalent.*

(i) $\vec{y} \in Core(v)$, *i.e.,* $\vec{y}(N) = v(N)$ *and* $\vec{y}(S) \geq v(S)$ *for all* $S \subseteq N$, $S \neq \emptyset$

(ii) $\vec{y}(N) = v(N)$ *and* $\pi^v(\vec{y}) = 0$

(iii) $\vec{y}(N) = v(N)$ *and* $\vec{y} \leq \vec{b}^v$, *i.e.,* $y_i \leq b_i^v$ *for all* $i \in N$

In the remainder of this section, we switch from 1-convex to 2-convex n-person games.

Theorem 4.5.18. *(D. Hou, T.S.H. Driessen, A. Meseguer-Artola,* **B. Mosoni** *[78]) Let the n-person TU game $v : \mathcal{P}(N) \to \mathbb{R}$ be 2-convex. Then its indirect function $\pi^v : \mathbb{R}^N \to \mathbb{R}$ satisfies the following properties:*

(i) $\pi^v(\vec{y}) = \max[0, \quad v(N) - \sum_{k \in N} y_k, \quad (v(\{i\}) - y_i)_{i \in N}]$ *for all* $\vec{y} \in \mathbb{R}^N$ *with* $\vec{y} \leq \vec{b}^v$.

(ii) $\pi^v(\vec{y}) = \max[0, \quad v(N \backslash \{\ell\}) - \sum_{k \in N \backslash \{\ell\}} y_k] = \max[0, \quad v(N) - \sum_{k \in N} y_k + y_\ell - b_\ell^v]$ *for all* $\vec{y} = (y_k)_{k \in N} \in \mathbb{R}^N$ *such that there exists a unique* $\ell \in N$ *with* $y_\ell > b_\ell^v \geq v(\{\ell\})$ *and* $v(\{i\}) \leq y_i \leq b_i^v$ *for all* $i \in N$, $i \neq \ell$.

(iii) $\pi^v(\vec{y}) = \max[v(N) - \sum_{k \in N} y_k, \quad v(\{j\}) - y_j]$ *for all* $\vec{y} = (y_k)_{k \in N} \in \mathbb{R}^N$ *such that there exists a unique* $j \in N$ *with* $y_j < v(\{j\}) \leq b_j^v$ *and* $v(\{i\}) \leq y_i \leq b_i^v$ *for all* $i \in N$, $i \neq j$.

(iv) $\pi^v(\vec{y}) = \max[v(N \backslash \{\ell\}) - \sum_{k \in N \backslash \{\ell\}} y_k, \quad v(\{j\}) - y_j] = \max[v(N) - \sum_{k \in N} y_k + y_\ell - b_\ell^v, \quad v(\{j\}) - y_j]$ *for all* $\vec{y} = (y_k)_{k \in N} \in \mathbb{R}^N$ *such that there exist unique* $j, \ell \in N$ *with* $y_\ell > b_\ell^v \geq v(\{\ell\})$, $y_i \leq b_i^v$ *for all* $i \in N$, $i \neq \ell$, *and* $y_j < v(\{j\}) \leq b_j^v$, $y_i \geq v(\{i\})$ *for all* $i \in N$, $i \neq j$.

Corollary 4.5.19. *(D. Hou, T.S.H. Driessen, A. Meseguer-Artola,* **B. Mosoni** *[78]) For every 2-convex n-person TU game* $v : \mathcal{P}(N) \to \mathbb{R}$*, the following three statements concerning a payoff vector* $\vec{y} = (y_k)_{k \in N} \in \mathbb{R}^N$ *are equivalent.*

(i) $\vec{y} \in Core(v)$, *i.e.*, $\vec{y}(N) = v(N)$ *and* $\vec{y}(S) \geq v(S)$ *for all* $S \subseteq N$, $S \neq \emptyset$

(ii) $\vec{y}(N) = v(N)$ *and* $\pi^v(\vec{y}) = 0$

(iii) $\vec{y}(N) = v(N)$ *and* $v(\{i\}) \leq y_i \leq b_i^v$ *for all* $i \in N$

Theorem 4.5.20. *(D. Hou, T.S.H. Driessen, A. Meseguer-Artola,* **B. Mosoni** *[78]) Let the n-person TU game* $v : \mathcal{P}(N) \to \mathbb{R}$ *be a big boss game, say player 1 is the big boss. Then its indirect function* $\pi^v : \mathbb{R}^N \to \mathbb{R}$ *satisfies the following properties:*

(i) $\pi^v(\vec{y}) = \max[0, \quad v(N) - \sum_{k \in N} y_k]$ *for all* $\vec{y} = (y_k)_{k \in N} \in \mathbb{R}^N$ *with* $0 \leq y_i \leq b_i^v$ *for all* $i \in N \backslash \{1\}$.

(ii) $\pi^v(\vec{y}) = \max[0, \quad v(N \backslash \{\ell\}) - \sum_{k \in N \backslash \{\ell\}} y_k] = \max[0, \quad v(N) - \sum_{k \in N} y_k + y_\ell - b_\ell^v]$ *for all* $\vec{y} = (y_k)_{k \in N} \in \mathbb{R}^N$ *such that there exists a unique* $\ell \in N \backslash \{1\}$ *with* $y_\ell > b_\ell^v \geq 0$ *and* $0 \leq y_i \leq b_i^v$ *for all* $i \in N \backslash \{1, \ell\}$.

(iii) $\pi^v(\vec{y}) = \max[-y_\ell, \quad v(N) - \sum_{k\in N} y_k]$ *for all* $\vec{y} = (y_k)_{k\in N} \in \mathbb{R}^N$ *such that there exists a unique* $\ell \in N\backslash\{1\}$ *with* $y_\ell < 0 \leq b_\ell^v$ *and* $0 \leq y_i \leq b_i^v$ *for all* $i \in N\backslash\{1,\ell\}$.

(iv) $\pi^v(\vec{y}) = \max[-y_j, \quad v(N\backslash\{\ell\}) - \sum_{k\in N\backslash\{\ell\}} y_k] = \max[-y_j, \quad v(N) - \sum_{k\in N} y_k + y_\ell - b_\ell^v]$ *for all* $\vec{y} = (y_k)_{k\in N} \in \mathbb{R}^N$ *such that there exist unique* $j, \ell \in N\backslash\{1\}$ *with* $y_\ell > b_\ell^v \geq 0$, $y_j < 0 \leq b_j^v$, *and* $0 \leq y_i \leq b_i^v$ *for all* $i \in N\backslash\{1,j,\ell\}$.

Corollary 4.5.21. *(D. Hou, T.S.H. Driessen, A. Meseguer-Artola,* **B. Mosoni** *[78]) For every n-person big boss game* $v : \mathcal{P}(N) \to \mathbb{R}$, *with player* 1 *as the big boss, the following three statements concerning a payoff vector* $\vec{y} = (y_k)_{k\in N} \in \mathbb{R}^N$ *are equivalent.*

(i) $\vec{y} \in Core(v)$, *i.e.,* $\vec{y}(N) = v(N)$ *and* $\vec{y}(S) \geq v(S)$ *for all* $S \subseteq N$, $S \neq \emptyset$

(ii) $\vec{y}(N) = v(N)$ *and* $\pi^v(\vec{y}) = 0$

(iii) $\vec{y}(N) = v(N)$ *and* $0 \leq y_i \leq b_i^v$ *for all* $i \in N\backslash\{1\}$

Theorem 4.5.22. *(D. Hou, T.S.H. Driessen, A. Meseguer-Artola,* **B. Mosoni** *[78]) Let the n-person TU game* $v : \mathcal{P}(N) \to \mathbb{R}$ *be a clan game, say coalition* $T \subseteq N$ *with at least two players is the clan. Then its indirect function* $\pi^v : \mathbb{R}^N \to \mathbb{R}$ *satisfies the following properties:*

(i) $\pi^v(\vec{y}) = \max[0, \quad v(N) - \sum_{k\in N} y_k]$ *for all* $\vec{y} = (y_k)_{k\in N} \in \mathbb{R}^N$ *with* $y_i \geq 0$ *for all* $i \in N$ *and* $y_i \leq b_i^v$ *for all* $i \in N\backslash T$.

(ii) $\pi^v(\vec{y}) = \max[0, \quad v(N\backslash\{\ell\}) - \sum_{k\in N\backslash\{\ell\}} y_k] = \max[0, \quad v(N) - \sum_{k\in N} y_k + y_\ell - b_\ell^v]$ *for all* $\vec{y} = (y_k)_{k\in N} \in \mathbb{R}^N$ *such that there exists a unique* $\ell \in N\backslash T$ *with* $y_\ell > b_\ell^v \geq 0$, $y_i \leq b_i^v$ *for all* $i \in N\backslash T$, $i \neq \ell$, *and* $y_i \geq 0$ *for all* $i \in N$.

(iii) $\pi^v(\vec{y}) = \max[-y_\ell, \quad v(N) - \sum_{k\in N} y_k]$ *for all* $\vec{y} = (y_k)_{k\in N} \in \mathbb{R}^N$ *such that there exists a unique* $\ell \in N$ *with* $y_\ell < 0$, $y_i \geq 0$ *for all* $i \in N\backslash\{\ell\}$, *and* $y_i \leq b_i^v$ *for all* $i \in N\backslash T$.

(iv) $\pi^v(\vec{y}) = \max[-y_j, \quad v(N\backslash\{\ell\}) - \sum_{k\in N\backslash\{\ell\}} y_k] = \max[-y_j, \quad v(N) - \sum_{k\in N} y_k + y_\ell - b_\ell^v]$ *for all* $\vec{y} = (y_k)_{k\in N} \in \mathbb{R}^N$ *such that there exist unique* $j \in N$, $\ell \in N\backslash T$ *with* $y_j < 0$, $y_i \geq 0$ *for all* $i \in N\backslash\{j\}$, *and* $y_\ell > b_\ell^v \geq 0$, $y_i \leq b_i^v$ *for all* $i \in N\backslash T$, $i \neq \ell$.

Corollary 4.5.23. *(D. Hou, T.S.H. Driessen, A. Meseguer-Artola,* **B. Mosoni** *[78]) For every n-person clan game $v : \mathcal{P}(N) \to \mathbb{R}$, with coalition $T \subseteq N$ as the clan, the following three statements concerning a payoff vector $\vec{y} = (y_k)_{k \in N} \in \mathbb{R}^N$ are equivalent.*

(i) $\vec{y} \in Core(v)$, *i.e.,* $\vec{y}(N) = v(N)$ *and* $\vec{y}(S) \geq v(S)$ *for all* $S \subseteq N$, $S \neq \emptyset$

(ii) $\vec{y}(N) = v(N)$ *and* $\pi^v(\vec{y}) = 0$

(iii) $\vec{y}(N) = v(N)$ *and* $y_i \geq 0$ *for all* $i \in N$ *and* $y_i \leq b_i^v$ *for all* $i \in N \backslash T$

Finally, we remark that a geometrical characterization of a clan game, say with coalition $T \subseteq N$ as the clan.

Theorem 4.5.24. *(D. Hou, T.S.H. Driessen, A. Meseguer-Artola,* **B. Mosoni** *[78]) Let $v : \mathcal{P}(N) \to \mathbb{R}$ be an n-person TU game and $\vec{x} = (x_k)_{k \in N} \in \mathbb{R}^N$ satisfying the efficiency principle $\vec{x}(N) = v(N)$.*

(i) For every pair of players $i, j \in N$, $i \neq j$, the indirect function $\pi^v : \mathbb{R}^N \to \mathbb{R}$ satisfies $\pi^v(\vec{x}^{ij\delta}) = s_{ij}^v(\vec{x}) + \delta$, provided $\delta \geq 0$ is sufficiently large.

(ii) $\vec{x} \in \mathcal{K}^*(v)$ if and only if the evaluation of the pairwise bargaining ranges arising from $\vec{x}$ through the indirect function are in equilibrium, that is, for every pair of players $i, j \in N$, $i \neq j$, the indirect function satisfies $\pi^v(\vec{x}^{ij\delta}) = \pi^v(\vec{x}^{ji\delta})$ for δ sufficiently large.

Remark 4.5.25. *(D. Hou, T.S.H. Driessen, A. Meseguer-Artola,* **B. Mosoni** *[78]) Suppose the n-person TU game $v : \mathcal{P}(N) \to \mathbb{R}$ is 1-convex. For every payoff vector $\vec{x} = (x_k)_{k \in N} \in \mathbb{R}^N$ satisfying the efficiency principle $\vec{x}(N) = v(N)$ as well as $\vec{x} \leq \vec{b}^v$, and for every pair of players $i, j \in N$, $i \neq j$, the evaluation of the indirect function $\pi^v : \mathbb{R}^N \to \mathbb{R}$ at the tail of the bargaining range described by the corresponding modified payoff vector $\vec{x}^{ij\delta}$ is in accordance with Theorem 4.5.16(i)–(ii) dependent on the size of its j-th component $\vec{x}_j^{ij\delta} = x_j + \delta$ in comparison to player j-th marginal benefit b_j^v. From the explicit formula for the indirect function of 1-convex games, we conclude the following:*

$$\begin{aligned} \pi^v(\vec{x}^{ij\delta}) &= 0 \quad \textit{if} \quad x_j^{ij\delta} \leq b_j^v, \textit{ that is } \delta \leq b_j^v - x_j \\ \pi^v(\vec{x}^{ij\delta}) &= \max[0, \quad x_j^{ij\delta} - b_j^v] = x_j + \delta - b_j^v > 0 \quad \textit{otherwise} \end{aligned}$$

For sufficiently large δ, the equilibrium condition $\pi^v(\vec{x}^{ij\delta}) = \pi^v(\vec{x}^{ji\delta})$ is met if and only if $x_j + \delta - b_j^v = x_i + \delta - b_i^v$, that is $x_j - b_j^v = x_i - b_i^v$ for all $i \neq j$. Together with the efficiency principle $\vec{x}(N) = v(N)$, the unique solution of this system of linear equations is given by

$$x_i = b_i^v - \frac{\alpha}{n} \qquad \textit{for all } i \in N, \textit{ where} \qquad \alpha = \vec{b}^v(N) - v(N) \geq 0$$

The latter solution is known as the nucleolus and turns out to coincide with the gravity of the core being the convex hull of n extreme points of the form $\vec{b}^v - \alpha \cdot \vec{e}_i$, $i \in N$. Here $\{\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_n\}$ denotes the standard basis of $\mathbb{R}^n$.

Remark 4.5.26. *(D. Hou, T.S.H. Driessen, A. Meseguer-Artola,* **B. Mosoni** *[78]) Suppose the TU game $v : \mathcal{P}(N) \to \mathbb{R}$ is a big boss game, with player 1 as the big boss. For every payoff vector $\vec{x} = (x_k)_{k \in N} \in \mathbb{R}^N$ satisfying the efficiency principle $\vec{x}(N) = v(N)$ as well as $0 \leq x_k \leq b_k^v$ for all $k \in N \backslash \{1\}$, and for every pair of players $i, j \in N$, $i \neq j$, the evaluation of the indirect function $\pi^v : \mathbb{R}^N \to \mathbb{R}$ at the tail of the bargaining range described by the corresponding modified payoff vector $\vec{x}^{j\ell\delta}$ is in accordance with Theorem 4.5.20(i)–(iv) dependent on the size of its j-th component $\vec{x}_j^{j\ell\delta} = x_j - \delta$ in comparison to the zero level as well as its ℓ-th component $\vec{x}_\ell^{j\ell\delta} = x_\ell + \delta$ in comparison to player ℓ-th marginal benefit b_ℓ^v. From the explicit formula for the indirect function of big boss games, we conclude the following: for $\{j, \ell\} \subseteq N \backslash \{1\}$, and for $\delta \geq 0$ sufficiently large*

$$\begin{aligned}
\pi^v(\vec{x}^{j\ell\delta}) &= \max[-(x_j - \delta),\ (x_\ell + \delta) - b_\ell^v] = \delta - \min[x_j,\ b_\ell^v - x_\ell] \\
\pi^v(\vec{x}^{1\ell\delta}) &= \max[0,\ (x_\ell + \delta) - b_\ell^v] = \delta + x_\ell - b_\ell^v \\
\pi^v(\vec{x}^{\ell 1\delta}) &= \max[0,\ -(x_\ell - \delta)] = \delta - x_\ell
\end{aligned}$$

For all $\ell \in N \backslash \{1\}$ and sufficiently large δ, the equilibrium condition $\pi^v(\vec{x}^{1\ell\delta}) = \pi^v(\vec{x}^{\ell 1\delta})$ is met if and only if $x_\ell - b_\ell^v = -x_\ell$, that is $x_\ell = \frac{b_\ell^v}{2}$ for all $\ell \neq 1$. Further, the equilibrium condition $\pi^v(\vec{x}^{j\ell\delta}) = \pi^v(\vec{x}^{\ell j\delta})$ for any pair $\{j, \ell\} \subseteq N \backslash \{1\}$ is given by

$$\min[x_j,\ b_\ell^v - x_\ell] = \min[x_\ell,\ b_j^v - x_j] \textit{which equalities are satisfied.}$$

Remark 4.5.27. *(D. Hou, T.S.H. Driessen, A. Meseguer-Artola,* **B. Mosoni** *[78]) Suppose the TU game $v : \mathcal{P}(N) \to \mathbb{R}$ is a clan game, say coalition $T \subseteq N$ with at least two players is the clan. From the explicit formula for the indirect function of clan games, as presented in Theorem 4.5.22 (ii)–(iv), we conclude that, for $\delta \geq 0$ sufficiently large, the equilibrium condition*

$\pi^v(\vec{x}^{ij\delta}) = \pi^v(\vec{x}^{ji\delta})$ *reduces to the following system of equations:* $x_i = x_j$ *for all* $i, j \in T$*, and*

$$x_i = \min[b_i^v - x_i,\ x_j] \text{ whenever } i \notin T,\ j \in T$$

$$\min[b_j^v - x_j,\ x_i] = \min[b_i^v - x_i,\ x_j] \text{ whenever } i, j \notin T$$

In summary, the unique solution is a so-called constrained equal reward rule of the form $x_i = \lambda$ *for all* $i \in T$ *and* $x_i = \min[\lambda,\ \frac{b_i^v}{2}]$ *for all* $i \in N\backslash T$*, where the parameter* $\lambda \in \mathbb{R}$ *is determined by the efficiency condition* $\vec{x}(N) = v(N)$*.*

Remark 4.5.28. *(D. Hou, T.S.H. Driessen, A. Meseguer-Artola,* **B. Mosoni** *[78]) Suppose the* n*-person TU game* $v : \mathcal{P}(N) \to \mathbb{R}$ *is* 2*-convex. From the explicit formula for the indirect function of* 2*-convex* n*-person games, as presented in Theorem 4.5.18(iv), we conclude that, for* $\delta \geq 0$ *sufficiently large, the equilibrium condition* $\pi^v(\vec{x}^{j\ell\delta}) = \pi^v(\vec{x}^{\ell j\delta})$ *reduces to the following system of equations: for every pair of players* $j, \ell \in N$*,* $j \neq \ell$*,*

$$\min[b_\ell^v - x_\ell, \quad x_j - v(\{j\})] = \min[b_j^v - x_j, \quad x_\ell - v(\{\ell\})]$$

As shown in [67], the unique solution is of the parametric form $x_i = v(\{i\}) + \min[\mu, \quad b_i^v - \frac{v(\{i\})}{2}]$ *for all* $i \in N$*, where the parameter* $\mu \in \mathbb{R}$ *is determined by the efficiency condition* $\vec{x}(N) = v(N)$*.*

Chapter 5

The maximal monotonicity of the generalized sum of two maximal monotone operators

In what follows X, respectively Y will be Banach spaces, and X^*, respectively Y^* denote their dual spaces. Let $S : X \to 2^{X^*}$, respectively $T : Y \to 2^{Y^*}$ be two monotone operators. Moreover, consider the continuous, linear operator $A : X \to Y$, and let us denote by A^* its adjoint operator. Recall that the a generalized sum (see [130]), of the monotone operators S, respectively T is defined as

$$M : X \to 2^{X^*},\ M(x) = (S + A^*TA)(x).$$

Obviously, when $X = Y$ and $A \equiv id_X$, this sum collapses to the sum of the monotone operators, that is

$$M : X \to 2^{X^*},\ M(x) := (S + T)(x),$$

while in the case when $S(x) = 0$ for all $x \in X$, we obtain the composite operator

$$M : X \to 2^{X^*},\ M(x) = A^*TA(x).$$

Consider X a separated locally convex space and X^* its topological dual space.

5.1 Maximal monotone operators and representative functions

Consider further X a nontrivial Banach space, X^* its topological dual space and X^{**} its bidual space. A set-valued operator $S : X \to 2^{X^*}$ is said to be *monotone* if

$$\langle y^* - x^*, y - x\rangle \geq 0, \text{ whenever } y^* \in S(y) \text{ and } x^* \in S(x).$$

The monotone operator S is called *maximal monotone* if its graph

$$G(S) = \{(x, x^*) : x^* \in S(x)\} \subseteq X \times X^*$$

is not properly contained in the graph of any other monotone operator $S' : X \to 2^{X^*}$. For S we consider also its *domain* $D(S) = \{x \in X : S(x) \neq \emptyset\} = pr_X(G(S))$ and its *range* $R(S) = \cup_{x\in X} S(x) = pr_{X^*}(G(S))$.

An element $(x_0, x_0^*) \in X \times X^*$ is said to be *monotonically related* to the graph of S if

$$\langle y^* - x_0^*, y - x_0\rangle \geq 0 \text{ for all } (y, y^*) \in G(S).$$

One can show that a monotone operator S is maximal monotone if and only if the set of monotonically related elements to $G(S)$ is exactly $G(S)$.

To an arbitrary monotone operator $S : X \to 2^{X^*}$ we associate the *Fitzpatrick function* $\varphi_S : X \times X^* \to \overline{\mathbb{R}}$, defined by

$$\varphi_S(x, x^*) = \sup\{\langle y^*, x\rangle + \langle x^*, y\rangle - \langle y^*, y\rangle : y^* \in S(y)\},$$

which is obviously convex and strong-weak* lower semicontinuous (it is even weak-weak* lower semicontinuous) in the corresponding topology on $X \times X^*$. Introduced by Fitzpatrick in 1988 (see [72]) and rediscovered after some years in [44, 104], it proved to be very important in the theory of maximal monotone operators, revealing important connections between convex analysis and monotone operators (see [19,32], [40,41], [44,105], [129,130,140,144], [127,128,145,156] and the references therein).

Considering the function $c : X \times X^* \to \mathbb{R}$, $c(x, x^*) = \langle x^*, x\rangle$ for all $(x, x^*) \in X \times X^*$, we get the equality $\varphi_S(x, x^*) = c_S^*(x^*, x)$ for all $(x, x^*) \in X \times X^*$, where $c_S = c + \delta_{G(S)}$ and we are considering the natural injection $X \subseteq X^{**}$. The function $psi_S = cl_{\|\cdot\|\times\|\cdot\|_*}(co c_S)$, where the closure is taken in the strong topology of $X \times X^*$, is well-linked to the Fitzpatrick function. Its properties were intensively studied in reflexive Banach spaces in [129] and in general Banach spaces in [44]. Let us mention that on $X \times X^*$ we have $\psi_S^{*\top} = \varphi_S$ and, in the framework of reflexive Banach spaces the equality $\varphi_S^{*\top} = \psi_S$ holds (see [44, Remark 5.4]). Let us recall the most important properties of the Fitzpatrick function.

Lemma 5.1.29. *(*see [72]*) Let $S : X \to 2^{X^*}$ be a maximal monotone operator. Then* *(i)* $\varphi_S(x, x^*) \geq \langle x^*, x\rangle$ *for all* $(x, x^*) \in X \times X^*$,

(ii) $G(S) = \{(x, x^*) \in X \times X^* : \varphi_S(x, x^*) = \langle x^*, x\rangle\}$.

Motivated by these properties of the Fitzpatrick function, the notion of *representative function* of a monotone operator was introduced and studied in the literature.

Definition 5.1.30. *For* $S : X \to 2^{X^*}$ *a monotone operator, we call* representative function *of* S *a convex and lower semicontinuous function* $h_S : X \times X^* \to \overline{\mathbb{R}}$ *(in the strong topology of* $X \times X^*$*) fulfilling*

$$h_S \geq c \text{ and } G(S) \subseteq \{(x, x^*) \in X \times X^* : h_S(x, x^*) = \langle x^*, x\rangle\}.$$

We observe that if $G(S) \neq \emptyset$ (in particular if S is maximal monotone), then every representative function of S is proper. It follows immediately that the Fitzpatrick function associated to a maximal monotone operator is a representative function of the operator. The following proposition is a direct consequence of some results given in [44].

Proposition 5.1.31. *Let* $S : X \to 2^{X^*}$ *be a maximal monotone operator and* h_S *be a representative function of* S*. Then*

(i) $\varphi_S \leq h_S \leq \psi_S$,

(ii) the canonical restriction of $h_S^{*\top}$ *to* $X \times X^*$ *is also a representative function of* S,

(iii) $\{(x, x^*) \in X \times X^* : h_S(x, x^*) = \langle x^*, x\rangle\} = \{(x, x^*) \in X \times X^* : h_S^{*\top}(x, x^*) = \langle x^*, x\rangle\} = G(S)$.

Remark 5.1.32. *In many situations the representative functions are lower semicontinuous in the strong-weak* topology, as it is the case for example for the Fitzpatrick functions of monotone operators. As Proposition 5.1.31(ii) shows, for every representative function of a maximal monotone operator one obtains a corresponding representative function which is strong-weak* lower semicontinuous. Moreover, when* $S = \partial f$*, where* $f : X \to \overline{\mathbb{R}}$ *is a proper, convex and lower semicontinuous function, then the function* $(x, x^*) \mapsto f(x) + f^*(x^*)$*, which is a representative of* ∂f*, is lower semicontinuous in the strong-weak* topology. Hence, for* $S : X \to 2^{X^*}$ *a monotone operator, it is natural to consider also the subfamily of* $\mathcal{H}(S)$ *formed by those representative functions of* S *which are lower semicontinuous with respect to the strong-weak* topology of* $X \times X^*$*. Let us notice that in general this is a proper subfamily (cf. [146, Remark 1]), while in the setting of reflexive Banach spaces it coincides with* $\mathcal{H}(S)$.

Let us give the following maximality criteria valid in reflexive Banach spaces (cf. [43, Theorem 3.1] and [130, Proposition 2.1]; see also [141] for other maximality criteria in reflexive spaces). We refer to [106, Theorem 4.2] for a generalization of the next result to arbitrary Banach spaces.

Theorem 5.1.33. *(*cf. [43, 130]*) Let X be a reflexive Banach space and $f : X \times X^* \to \overline{\mathbb{R}}$ a proper, convex and lower semicontinuous function such that $f \geq c$. Then the operator whose graph is the set $\{(x, x^*) \in X \times X^* : f(x, x^*) = \langle x^*, x\rangle\}$ is maximal monotone if and only if $f^{*\top}\big|_{X\times X^*} \geq c$.*

The following particular class of maximal monotone operators has been recently introduced in [106], being also studied in [147].

Definition 5.1.34. *An operator $S : X \to 2^{X^*}$ is said to be* strongly-representable *whenever there exists a proper, convex and strong lower semicontinuous function $h : X \times X^* \to \overline{\mathbb{R}}$ such that*

$$h \geq c, h^*(x^*, x^{**}) \geq \langle x^{**}, x^*\rangle \forall (x^*, x^{**}) \in X^* \times X^{**}$$

and

$$G(S) = \{(x, x^*) \in X \times X^* : h(x, x^*) = \langle x^*, x\rangle\}.$$

In this case h is called a strong-representative *of S.*

We will need the following result (see [106, Theorem 4.2]).

Theorem 5.1.35. *Let X be a nonzero Banach space and $h : X \times X^* \to \overline{\mathbb{R}}$ a proper, convex and lower semicontinuous function such that $h \geq c$ and $h^*(x^*, x^{**}) \geq \langle x^{**}, x^*\rangle$ for all $(x^*, x^{**}) \in X^* \times X^{**}$. Then the operator whose graph is the set $\{(x, x^*) \in X \times X^* : h(x, x^*) = \langle x^*, x\rangle\}$ is maximal monotone and it holds $\{(x, x^*) \in X \times X^* : h(x, x^*) = \langle x^*, x\rangle\} = \{(x, x^*) \in X \times X^* : h^*(x^*, x) = \langle x^*, x\rangle\}$.*

Hence, if $S : X \to 2^{X^*}$ is strongly-representable, then S is maximal monotone (see also [147, Theorem 8]), and φ_S is a strong-representative of S.

Definition 5.1.36. *(*see [75]*) Gossez's monotone closure of a maximal monotone operator $S : X \to 2^{X^*}$ is $\overline{S} : X^{**} \to 2^{X^*}$,*

$$G(\overline{S}) = \{(x^{**}, x^*) \in X^{**} \times X^* : \langle x^* - y^*, x^{**} - y\rangle \geq 0, \forall (y, y^*) \in G(S)\}.$$

A maximal monotone operator $S : X \to 2^{X^*}$ is of Gossez type (D) if for any $(x^{**}, x^*) \in G(\overline{S})$, there exists a bounded net $\{(x_\alpha, x^*_\alpha)\}_{\alpha\in\mathfrak{I}} \subseteq G(S)$ which converges to (x^{**}, x^*) in the $w^* \times \|\cdot\|$ topology of $X^{**} \times X^*$.

In [142] Simons introduced a new class of maximal monotone operators, called operators of negative infimum type (NI).

Definition 5.1.37. *(*see [142]*) A maximal monotone operator $S : X \to 2^{X^*}$ is of Simons type (NI) if*

$$\inf_{(y,y^*)\in G(S)} \langle y^* - x^*, y - x^{**}\rangle \geq 0, \ (\forall)(x^*, x^{**}) \in X^* \times X^{**}.$$

5.2 The stable strong duality and a generalized bivariate infimal convolution formula

Let X and Y be real separated locally convex spaces, with their topological duals X^* and Y^*, respectively.

Theorem 5.2.38. *(***B. Burjan-Mosoni (Mosoni)***, S. László [47]) Let $f : X \to \overline{\mathbb{R}}$ and $g : Y \to \overline{\mathbb{R}}$ be proper, convex and lower semicontinuous functions, let $A : X \to Y$ be a linear and continuous operator, and $A^* : Y^* \to X^*$ be its adjoint operator. Assume that $dom(f) \cap A^{-1}(dom(g)) \neq \emptyset$.*

(a) Let U be a nonempty subset of X^*. The following statements are equivalent:

(i) $\sup_{x\in X}\{\langle x^*, x\rangle - (f + g\circ A)(x)\} = \min_{y^*\in Y^*}\{f^*(x^* - A^*y^*) + g^*(y^*)\}$ for all $x^* \in U$.

(ii) The set $\{(x^* + A^*y^*, r) : f^*(x^*) + g^*(y^*) \leq r\}$ is closed regarding to $U \times \mathbb{R}$ in $(X^*, w^*) \times \mathbb{R}$ topology.

(b) If X and Y are Fréchet spaces and

$$0 \in {}^{ic}(dom(g) - A(dom(f))),$$

then the statements (i) and (ii) are valid for every $U \subseteq X^*$.

Remark 5.2.39. According to Proposition 4 from [153], if X and Y are Fréchet spaces, we have

$${}^{ic}(pr_Y(dom\Phi_A)) = ri(pr_Y(dom\Phi_A)),$$

or, equivalently

$${}^{ic}(dom(g) - A(dom(f))) = ri(dom(g) - A(dom(f))).$$

Remark 5.2.40. (**B. Burjan-Mosoni (Mosoni)**, S. László [47])Observe that if X and Y are Fréchet spaces, then the condition the set $\{(x^*+A^*y^*, r) : f^*(x^*) + g^*(y^*) \leq r\}$ is closed in $(X^*, w^*) \times \mathbb{R}$ topology is weaker than the condition $0 \in {}^{ic}(dom(g) - A(dom(f)))$.

In what follows, let X and Y be two normed spaces, with their dual X^* and Y^*, and consider the proper, convex and lower semicontinuous functions $f : X \times X^* \to \overline{\mathbb{R}}$ and $g : Y \times Y^* \to \overline{\mathbb{R}}$. Assume, that the dual spaces of $X \times X^*$ and $Y \times Y^*$ respectively, $X^* \times X^{**}$ and $Y^* \times Y^{**}$ respectively, are endowed with the w^* topology. Moreover, let $A : X \to Y$ be a linear and continuous operator and $A^* : Y^* \to X^*$, respectively $A^{**} : X^{**} \to Y^{**}$ be its adjoint, respectively its biadjoint operator.

Consider the following generalized inf-convolution formulas, $f \triangle_2^A g : X \times X^* \to \overline{\mathbb{R}}$

$$(f \triangle_2^A g)(x, x^*) = \inf\{f(x, x^* - A^*y^*) + g(Ax, y^*) : \, y^* \in Y^*\},$$

respectively, $f^* \triangle_1^A g^* : X^* \times X^{**} \to \overline{\mathbb{R}}$, $(f^* \triangle_1^A g^*)(x^*, x^{**})$

$$= \inf\{f^*(x^* - A^*y^*, x^{**}) + g^*(y^*, A^{**}x^{**}) : \, y^* \in Y^*\}.$$

The above formulas were intensively studied by Voisei and Zalinescu in [147], Simons and Zalinescu in [144], and Simons in [143]. However, they provided interior-point regularity conditions only, that ensures that $(f \triangle_2^A g)^*(x^*, x^{**}) = (f^* \triangle_1^A g^*)(x^*, x^{**})$ and $f^* \triangle_1^A g^*$ is exact for every $(x^*, x^{**}) \in X^* \times X^{**}$. Obviously, when $A \equiv id_X$, $X = Y$ we obtain $f^* \Box_1 g^*$ and $f \Box_2 g$, respectively, (see, for instance, [37, 140, 144, 147]), that is

$$(f^* \Box_1 g^*)(x^*, x^{**}) = \inf\{f^*(x^* - y^*, x^{**}) + g^*(y^*, x^{**}) : \, y^* \in X^*\},$$

respectively, $(f \Box_2 g)(x, x^*)$

$$= \inf\{f(x, x^* - y^*) + g(x, y^*) : \, y^* \in X^*\}.$$

The following result provides a closedness type regularity condition that not only ensures that $(f \triangle_2^A g)^*(x^*, x^{**}) = (f^* \triangle_1^A g^*)(x^*, x^{**})$ and $f^* \triangle_1^A g^*$ is exact for every $(x^*, x^{**}) \in X^* \times X^{**}$, but also is equivalent to it.

Theorem 5.2.41. *(***B. Burjan-Mosoni (Mosoni)***, S. László [47])*
Assume that $A(pr_X(dom(f))) \cap (pr_Y(dom(g))) \neq \emptyset$.

a) The following statements are equivalent:

(i) The set $\{(x^* + A^*y^*, x^{**}, y^{**}, r) : f^*(x^*, x^{**}) + g^*(y^*, y^{**}) \leq r\}$ *is closed regarding the set* $X^* \times \Delta_{X^{**}}^{A^{**}} \times \mathbb{R}$ *in the* $(X^*, w^*) \times (X^{**}, w^*) \times (Y^{**}, w^*) \times \mathbb{R}$ *topology, where* $\Delta_{X^{**}}^{A^{**}} = \{(x^{**}, A^{**}x^{**}) : x^{**} \in X^{**}\}$.

(ii) $(f\triangle_2^A g)^*(x^*,x^{**}) = (f^*\triangle_1^A g^*)(x^*,x^{**})$ *and* $f^*\triangle_1^A g^*$ *is exact (that is, the infimum in the definition of* $f^*\triangle_1^A g^*$ *is attained) for every* $(x^*,x^{**}) \in X^* \times X^{**}$.

b) If

$$0 \in {}^{ic}(pr_Y dom(g) - A(pr_X dom(f)))$$

then the statements (i) and (ii) are true.

Remark 5.2.42. *(***B. Burjan-Mosoni (Mosoni)***, S. László [47])*

In the hypotheses of Theorem 5.2.41 and by keeping the notations used in its proof, according to Remark 5.2.39, we have

$${}^{ic}(dom(G) - N(dom(F))) = ri(dom(G) - N(dom(F))),$$

which is equivalent to

$$\begin{aligned} &{}^{ic}(pr_Y dom(g) - A(pr_X dom(f))) \\ &= ri(pr_Y dom(g) - A(pr_X dom(f))). \end{aligned}$$

By taking $X = Y$ and $A \equiv id_X$ in Theorem 5.2.41 we obtain the following result, (see also [37]).

Corollary 5.2.43. *(***B. Burjan-Mosoni (Mosoni)***, S. László [47])*

Assume that $pr_X(dom(f) \cap (pr_X(dom(g))) \neq \emptyset$.

a) The following statements are equivalent:

(i) The set $\{(u^*+v^*, u^{**}, v^{**}, r) : f^*(u^*,u^{**}) + g^*(v^*,v^{**}) \le r\}$ is closed regarding the set $X^* \times \Delta_{X^{**}} \times \mathbb{R}$ in the $(X^*,w^*)\times(X^{**},w^*)\times(X^{**},w^*)\times \mathbb{R}$ topology, where $\Delta_{X^{**}} = \{(x^{**},x^{**}) : x^{**} \in X^{**}\}$.

(ii) $(f\Box_2 g)^*(x^*,x^{**}) = (f^*\Box_1 g^*)(x^*,x^{**})$ and $f^*\Box_1 g^*$ is exact for every $(x^*,x^{**}) \in X^* \times X^{**}$.

b) If

$$\begin{aligned} 0 &\in {}^{ic}(pr_X dom(g) - pr_X dom(f)) \\ &= ri(pr_X dom(g) - pr_X dom(f)) \end{aligned}$$

then the statements (i) and (ii) are true.

Let now X and Y be a reflexive Banach spaces. Then, Theorem 5.2.41 becomes.

Corollary 5.2.44. *(***B. Burjan-Mosoni (Mosoni), S. László [47]***)*
Consider the proper, convex and lower semicontinuous functions $f : X \times X^* \to \overline{\mathbb{R}}$ and $g : Y \times Y^* \to \overline{\mathbb{R}}$. If $A(pr_X(dom(f))) \cap (pr_Y(dom(g))) \neq \emptyset$ then the following conditions are equivalent.

(i) The set $\{(x^* + A^*y^*, x, y, r) : f^*(x^*, x) + g^*(y^*, y) \leq r\}$ is closed regarding the set $X^* \times \Delta_X^A \times \mathbb{R}$ in the $(X^*, \|\cdot\|_*) \times (X, \|\cdot\|) \times (Y, \|\cdot\|) \times \mathbb{R}$ topology, where $\Delta_X^A = \{(x, Ax) : x \in X\}$.

(ii) $(f \triangle_2^A g)^*(x^*, x) = (f^* \triangle_1^A g^*)(x^*, x)$ and $f^* \triangle_1^A g^*$ is exact for every $(x^*, x) \in X^* \times X$.

Concerning on the formula $\triangle_1^A$ we are able to establish a similar result to Theorem 5.2.41 only in a reflexive Banach space context. In what follows we assume that X, respectively Y are reflexive Banach spaces, with their biduals identified with X, respectively Y. In this case we have $f^* \triangle_1^A g^* : X^* \times X \to \overline{\mathbb{R}}$,

$$(f^* \triangle_1^A g^*)(x^*, x) = \inf\{f^*(x^* - A^*y^*, x) + g^*(y^*, Ax) : \ y^* \in Y^*\}.$$

Theorem 5.2.45. *(***B. Burjan-Mosoni (Mosoni)***, S. László [47]) Assume that $A(pr_X(dom(f^*))) \cap (pr_Y(dom(g^*))) \neq \emptyset$.*

a) The following statements are equivalent:

(i) The set $\{(x^* + A^*y^*, x, y, r) : f(x, x^*) + g(y, y^*) \leq r\}$ is closed regarding the set $X^* \times \Delta_X^A \times \mathbb{R}$ in the $(X^*, \|\cdot\|_*) \times (X, \|\cdot\|) \times (Y, \|\cdot\|) \times \mathbb{R}$ topology, where $\Delta_X^A = \{(x, Ax) : x \in X\}$.

(ii) $(f^* \triangle_1^A g^*)^*(x, x^*) = (f \triangle_2^A g)(x, x^*)$ and $(f \triangle_2^A g)$ is exact (that is, the infimum in the definition of $f \triangle_2^A g$ is attained) for every $(x, x^*) \in X \times X^*$.

b) If

$$0 \in {}^{ic}(pr_Y dom(g^*) - A(pr_X dom(f^*)))$$

then the statements (i) and (ii) are true.

Remark 5.2.46. *(***B. Burjan-Mosoni (Mosoni)***, S. László [47])*In the hypotheses of Theorem 5.2.45 and by keeping the notations used in its proof, according to Remark 5.2.39, we have

$${}^{ic}(dom(G) - N(dom(F))) = ri(dom(G) - N(dom(F))),$$

which is equivalent to

$$\begin{aligned} &{}^{ic}(pr_Y dom(g^*) - A(pr_X dom(f^*))) \\ &= ri(pr_Y dom(g^*) - A(pr_X dom(f^*))). \end{aligned}$$

By taking $X = Y$ and $A \equiv id_X$ in Theorem 5.2.45 we obtain the following result.

Corollary 5.2.47. *(***B. Burjan-Mosoni (Mosoni)***, S. László [47])*
Assume that $pr_X(dom(f^*)) \cap pr_Y(dom(g^*)) \neq \emptyset$.

a) The following statements are equivalent:

(i) The set $\{(x^*+x^*, x, x, r) : f(x, x^*)+g(y, y^*) \leq r\}$ is closed regarding the set $X^* \times \Delta_X \times \mathbb{R}$ in the $(X^*, \|\cdot\|_*) \times (X, \|\cdot\|) \times (X, \|\cdot\|) \times \mathbb{R}$ topology, where $\Delta_X = \{(x, x) : x \in X\}$.

(ii) $(f^*\Box_1 g^*)^*(x, x^*) = (f\Box_2 g)(x, x^*)$ and $f\Box_2 g$ is exact for every $(x, x^*) \in X \times X^*$.

b) If

$$\begin{aligned} 0 &\in {}^{ic}(pr_X dom(g^*) - pr_X dom(f^*)) \\ &= ri(pr_X dom(g^*) - pr_X dom(f^*)) \end{aligned}$$

then the statements (i) and (ii) are true.

5.3 The maximal monotonicity of the operator $S + A^*TA$

In what follows X, respectively Y will be Banach spaces, X^*, respectively Y^* denote their dual spaces, X^{**}, respectively Y^{**} denote their bidual spaces. Consider the monotone operators $S : X \to 2^{X^*}$ and $T : Y \to 2^{Y^*}$ and let $A : X \to Y$ be a linear and continuous operator, and A^* its adjoint operator. A well known generalized sum involving S and T is defined as follows:

$$M : X \to 2^{X^*},\ M := S + A^*TA.$$

Obviously, when $X = Y$, $A \equiv id_X$, then M becomes the well known sum of the operators S and T, that is $M := S + T$, while in the case when $Sx = 0$, for all $x \in X$, M becomes the composition A^*TA.

In what follows we give some sufficient conditions which ensure the maximal monotonicity of $S + A^*TA$, where S, respectively T are maximal monotone operators of Gossez type (D).

Theorem 5.3.48. *(***B. Burjan-Mosoni (Mosoni)***, S. László [47]) Consider $A : X \to Y$ a linear and continuous operator and let us denote by A^* its adjoint operator, and by A^{**} its biadjoint operator. Let $S : X \to 2^{X^*}$ and $T : Y \to 2^{Y^*}$ be two strongly-representable monotone operators with strong* *representative functions h_S and h_T respectively, such that $A(pr_X(dom(h_S))) \cap (pr_Y(dom(h_T))) \neq \emptyset$. Consider the function $h : X \times X^* \to \overline{\mathbb{R}}$, $h(x, x^*) = cl_{\|\cdot\| \times \|\cdot\|_*}(h_S \triangle_2^A h_T)(x, x^*)$. Assume that one of the following conditions is fulfilled.*

(a) $0 \in {}^{ic}(pr_Y dom(h_T) - A(pr_X dom(h_S)))$;

(b) the set $\{(x^* + A^*y^*, x^{**}, y^{**}, r) : h_S^*(x^*, x^{**}) + h_T^*(y^*, y^{**}) \leq r\}$ *is closed regarding the set* $X^* \times \Delta_{X^{**}}^{A^{**}} \times \mathbb{R}$ *in the* $(X^*, w^*) \times (X^{**}, w^*) \times (Y^{**}, w^*) \times \mathbb{R}$ *topology, where* $\Delta_{X^{**}}^{A^{**}} = \{(x^{**}, A^{**}x^{**}) : x^{**} \in X^{**}\}$.

Then h is a strong representative function of $S + A^*TA$ and $S + A^*TA$ is a strongly-representable monotone operator.

Assume now, that X and Y are reflexive Banach spaces. Then the following result holds.

Theorem 5.3.49. *(***B. Burjan-Mosoni (Mosoni)***, S. László [47]) Consider $A : X \to Y$ a linear and continuous operator and let us denote by A^* its adjoint operator. Let $S : X \to 2^{X^*}$ and $T : Y \to 2^{Y^*}$ be two maximal monotone operators with representative functions h_S and h_T respectively, such that $A(pr_X(dom(h_S^*))) \cap (pr_Y(dom(h_T^*))) \neq \emptyset$. Consider the function $h : X \times X^* \to \overline{\mathbb{R}}$, $h(x, x^*) = (h_S^* \triangle_1^A h_T^*)^*(x, x^*)$. Assume that one of the following conditions is fulfilled.*

(a) $0 \in {}^{ic}(pr_Y dom(h_T^*) - A(pr_X dom(h_S^*)))$;

(b) the set $\{(x^* + A^*y^*, x, y, r) : h_S(x, x^*) + h_T(y, y^*) \leq r\}$ *is closed regarding the set* $X^* \times \Delta_X^A \times \mathbb{R}$ *in the* $(X^*, \| \cdot \|_*) \times (X, \| \cdot \|) \times (Y, \| \cdot \|) \times \mathbb{R}$ *topology.*

Then h is a representative function of $S + A^*TA$ and $S + A^*TA$ is a maximal monotone operator.

Let us mention that the results from this section were partially established by Simons in [143], Voisei and Zalinescu in [147, 148].

5.4 Particular cases

Considering $X = Y$ and $A \equiv id_X$ the generalized sum $S + A^*TA$ becomes $S + T$, and $\triangle_1^A$, respectively $\triangle_2^A$ become $\Box_1$, respectively $\Box_2$, hence from Theorem 5.3.48, we obtain the following:

Corollary 5.4.50. *(***B. Burjan-Mosoni (Mosoni)***, S. László [47])*

Let $S,T : X \to 2^{X^*}$ be two strongly-representable monotone operators, with strong representative functions h_S and h_T, such that

$$(pr_X(dom(h_S))) \cap (pr_Y(dom(h_T))) \neq \emptyset,$$

and consider the function $h : X \times X^* \to \overline{\mathbb{R}}$,

$$h(x,x^*) = cl_{\|\cdot\|\times\|\cdot\|_*}(h_S \Box_2 h_T)(x,x^*).$$

Assume that one of the following conditions is fulfilled.

(a) $0 \in {}^{ic}(pr_X dom(h_T) - pr_X dom(h_S))$;

(b) the set $\{(x^* + y^*, x^{**}, y^{**}, r) : h_S^*(x^*, x^{**}) + h_T^*(y^*, y^{**}) \leq r\}$ is closed regarding the set $X^* \times \Delta_{X^{**}} \times \mathbb{R}$ in the $(X^*, w^*) \times (X^{**}, w^*) \times (X^{**}, w^*) \times \mathbb{R}$ topology, where $\Delta_{X^{**}} = \{(x^{**}, x^{**}) : x^{**} \in X^{**}\}$.

Then h is a strong representative function of $S+T$, hence $S+T$ is a strongly-representable monotone operator.

Assume now, that X is a reflexive Banach space. Then according to Theorem 5.3.49, the following result holds.

Corollary 5.4.51. *(***B. Burjan-Mosoni (Mosoni)***, S. László [47]) Let $S,T : X \to 2^{X^*}$ be two maximal monotone operators with representative functions h_S and h_T respectively, such that $pr_X(dom(h_S^*)) \cap pr_X(dom(h_T^*)) \neq \emptyset$. Consider the function $h : X \times X^* \to \overline{\mathbb{R}}$*

$$h(x,x^*) = (h_S^* \Box_1 h_T^*)^*(x,x^*).$$

Assume that one of the following conditions is fulfilled.

(a) $0 \in {}^{ic}(pr_X dom(h_T^*) - pr_X dom(h_S^*))$*;*

(b) the set $\{(x^* + y^*, x, y, r) : h_S(x,x^*) + h_T(y,y^*) \leq r\}$ is closed regarding the set $X^* \times \Delta_X \times \mathbb{R}$ in the $(X^*, \|\cdot\|_*) \times (X, \|\cdot\|) \times (X, \|\cdot\|) \times \mathbb{R}$ topology.

Then h is a representative function of $S+T$ and $S+T$ is a maximal monotone operator.

Let us mention that the above result was partially obtained also in [37]. For the second particular instance assume that $S : X \to 2^{X^*}$ is the multivalued operator with $G(S) = X \times \{0\}$, which is obviously a strongly-representable operator. Its extension to the bidual, $\overline{S} : X^{**} \to 2^{X^*}$, fulfills $G(\overline{S}) = X \times \{0\}$. Since $\varphi_S = \psi_S = \delta_{X\times\{0\}}$, by Proposition 5.1.31 it follows that the only representative function of S is $h_S = \delta_{X\times\{0\}}$. Since $h_S^* = \delta_{\{0\}\times X^{**}}$, h_S is actually a strong representative function of S.

Having $h_T : Y \times Y^* \to \overline{\mathbb{R}}$ a representative function T, the extended infimal convolutions $h_S \triangle_2^A h_T$ and $h_S^* \triangle_2^A h_T^*$ of h_S and h_T become in this situation

$$h_T^A : X \times X^* \to \overline{\mathbb{R}}, h_T^A(x, x^*) = \inf\{h_T(Ax, v^*) : v^* \in Y^*, A^*v^* = x^*\}$$

and $h_T^{*A} : X^* \times X^{**} \to \overline{\mathbb{R}}$,

$$h_T^{*A}(x^*, x^{**}) = \inf\{h_T^*(v^*, A^{**}x^{**}) : v^* \in Y^*, A^*v^* = x^*\},$$

respectively.

Noticing that

$$A(pr_X(domh_S)) - pr_Y(domh_T) = im\ A - pr_Y(domh_T)$$

, Theorem 5.3.48 gives rise to the following result.

Corollary 5.4.52. *(***B. Burjan-Mosoni (Mosoni)***, S. László [47]) Let $T : Y \to 2^{Y^*}$ be a strongly-representable monotone operator with strong representative function h_T and $A : X \to Y$ a linear continuous mapping such that $im\ A \cap pr_Y(domh_T) \neq \emptyset$. Assume that one of the following conditions is fulfilled:*

(a) $0 \in {}^{ic}(im\ A - pr_Y(domh_T))$;

(b) the set $\{(A^*v^*, v^{**}, r) : r \in \mathbb{R}, h_T^*(v^*, v^{**}) \leq r\}$ *is closed regarding* $X^* \times im\ A^{**} \times \mathbb{R}$ *in* $(X^*, w^*) \times (Y^{**}, w^*) \times \mathbb{R}$ *topology.*

Then the function $h : X \times X^* \to \overline{\mathbb{R}}$, $h(x, x^*) = cl_{\|\cdot\|\times\|\cdot\|_*} h_T^A(x, x^*)$, is a strong representative function of A^*TA and A^*TA is a strongly-representable monotone operator.

Assume now, that X and Y are reflexive Banach spaces. Then according to Theorem 5.3.49, the following result holds.

Corollary 5.4.53. *(***B. Burjan-Mosoni (Mosoni)***, S. László [47]) Let $T : Y \to 2^{Y^*}$ be a maximal monotone operators with representative function h_T and $A : X \to Y$ a linear continuous mapping such that $im\ A \cap pr_Y(domh_T) \neq \emptyset$. Assume that one of the following conditions is fulfilled:*

(a) $0 \in {}^{ic}(im\ A - pr_Y(dom h_T))$;

(b) *the set* $\{(A^*v^*, v, r) : r \in \mathbb{R}, h_T(v, v^*) \leq r\}$ *is closed regarding* $X^* \times im\ A \times \mathbb{R}$ *in* $(X^*, |\cdot\|_*) \times (Y, |\cdot\|) \times \mathbb{R}$ *topology.*

Then the function $h : X \times X^* \to \overline{\mathbb{R}}$, $h(x, x^*) = h_T^A(x, x^*)$, *is a representative function of* A^*TA *and* A^*TA *is a maximal monotone operator.*

Let us mention that these results were partially also established by Voisei in [148].

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